Characterizing revealing and arbitrage-free financial markets

Lionel de Boisdeffre

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Characterizing revealing and arbitrage-free financial markets

Lionel De BOISDEFFRE

2016.42

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Characterizing revealing and arbitrage-free financial markets

Lionel de Boisdeffre[1]

(May 2016)

Abstract

Radner (1979) introduces a general equilibrium model of asymmetric information, where agents have a model "of how equilibrium prices are determined", without which they could not update their beliefs. Differently, De Boisdeffre (2016, [3]) shows that agents, having private anticipations and no price model, can still update their beliefs from observing trade on financial markets, until all arbitrage is precluded. Then, inferences consist in successively eliminating anticipations, which would grant an unlimited arbitrage, if realizable. Thus, in our model, agents learn from arbitrage opportunities on portfolios, as they would do on actual markets. This model is consistent with all kinds of assets and uncountably many forecasts. We now characterize arbitrage-free markets, and show that the information markets may reveal depends on the span of asset payoffs in agents' commonly expected states. We provide conditions, under which markets are non-informative or typically revealing.

Key words: anticipations, inferences, perfect foresight, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

[1] CATT-University of Pau, 1 Av. du Doyen Poplawski, 64000 Pau, France. Email: lionel.deboisdeffre@univ-pau.fr. Tel: 33(0)5.59.40.80.01. Fax: 33(0)5.59.40.80.10 and CES-University of Paris 1, 106-112 Bd. de l'Hôpital, 75013 Paris, France.

1 Introduction

Asymmetrically informed agents may infer information from observing prices or trade volumes on markets. Thus, in the Radner (1979) rational expectation setting "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". They may infer private information of other agents from comparing actual prices and price expectations with their theoretical values at a price revealing equilibrium. Yet, equilibrium may fail to exist. Existence in Radner's model only holds generically.

Hereafter, we drop both Radner's (1972, 1979) classical assumptions that agents have rational expectations and a perfect foresight of future prices. Instead, we consider a two-period model with uncountably many states, also called anticipations, expectations or forecasts. The state space captures the exogenous uncertainty, stemming from nature's play over future events, and also, typically, an endogenous uncertainty, resulting from the fact that agents' characteristics, forecasts and beliefs may be private. Agents' forecasts form idiosyncratic subsets of the state space.

Assets of any kind may be exchanged at the first period, whose payoffs, at the second, are state dependent. Starting from their idiosyncratic sets of anticipations, agents may update their beliefs from observing portfolios, and successively eliminating forecasts, that would grant an arbitrage, as in De Boisdeffre (2016, [3]).

The current paper studies the payoff structure of arbitrage-free markets, and the information markets may reveal. It shows this information depends on the span of asset payoffs in agents' commonly expected states. It provides conditions, under which markets are non-revealing or, typically, partially or fully revealing. In particular, when agents have perfect foresight, dropping Radner's (1979) assumption not only restores the full existence property of equilibrium, along De Boisdeffre (2007), but provides insights on the information agents reach.

The paper is organized as follows: Section 2 presents De Boisdeffre's (2016, [3]) model and its main results. Section 3 studies the information markets may convey. Section 4 concludes.

2 The model

We consider a pure-exchange economy with two periods ($t \in \{0, 1\}$) and finitely many agents, $i \in I := \{1, ..., m\}$, having uncertainty at the first period about which state, ω, will prevail tomorrow out of a state space, denoted by Ω, which stands for any open subset with cardinality of the continuum of a metric space (e.g., $\Omega :=]0, 1[$). States are also called forecasts, anticipations or expectations.

At $t = 0$, each agent, $i \in I$, receives a private information signal, in the form of a compact sub-set, Ω_i, of Ω, informing her that tomorrow's state will be in Ω_i. She then elects a probabilty distribution, π_i, on $(\Omega, \mathcal{B}(\Omega))$, called her belief, whose support is Ω_i. The information structure, $(\Omega_i, \pi_i)_{i \in I}$, is given throughout, with $\underline{\mathbf{\Omega}} := \cap_{i \in I} \Omega_i \neq \varnothing$. It may be refined from observing markets. Then, each agent, $i \in I$, reduces her forcasts to a subset of Ω_i, and consistently updates her belief.

Agents exchange finitely many assets, $j \in \mathcal{J} := \{1, ..., J\}$, at $t = 0$, whose cash payoffs, $v_j(\omega) \in \mathbb{R}$, are conditional on the occurence of a state $\omega \in \Omega$, at $t = 1$, and define a row vector, $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$. The mapping $\omega \in \Omega \mapsto V(\omega)$ is assumed to be continuous. Agents' positions in assets define portfolios, $z \in \mathbb{R}^J$. Given an asset price, $q \in \mathbb{R}^J$, a portofolio, $z \in \mathbb{R}^J$, costs $q \cdot z$ units of account at $t = 0$, and promises $V(\omega) \cdot z$ units tomorrow, in each state, $\omega \in \Omega$, if ω obtains. We now present arbitrage.

Definition 1 *A price, $q \in \mathbb{R}^J$, is said to be a common no-arbitrage price of $(\Omega_i)_{i\in I}$, or the structure $(\Omega_i)_{i\in I}$ to be q-arbitrage-free, if the following condition holds:*

(a) $\nexists(i,z) \in I \times \mathbb{R}^J : -q \cdot z \geqslant 0$ and $V(\omega) \cdot z \geqslant 0$, $\forall \omega \in \Omega_i$, with one strict inequality.

A structure, which admits a common no-arbitrage price, is called arbitrage-free.

We recall the following Claim, whose proof is given in De Boisdeffre (2016, [3]).

Claim 1 *Let $q \in \mathbb{R}^J$ be given. For each $i \in I$, we denote by $L_2^{++}(\pi_i)$ the set of mappings, $f : \Omega_i \to \mathbb{R}$, in the Riesz space $L_2(\pi_i)$, such that $f(\omega) > 0$ π_i-almost surely. The following statements are equivalent:*

(i) $q \in Q_c[(\Omega_i)]$, that is, (Ω_i) is q-arbitrage free;

(ii) $\forall i \in I$, $\exists f_i \in L_2^{++}(\pi_i)$, such that $q = \int_{\omega\in\Omega_i} V(\omega) f_i(\omega) d\pi_i(\omega)$;

Besides, (Ω_i) is arbitrage-free if and only if it meets the following AFAO Condition:

(iii) $\nexists(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0$, $V(\omega_i) \cdot z_i \geqslant 0$, $\forall (i, \omega_i) \in I \times \Omega_i$, one at least being strict.

De Boisdeffre (2016, [3]) shows that, if $(\Omega_i)_{i\in I}$ is arbitrage-free, agents cannot infer information and, otherwise, that they may always infer one unique coarsest arbitrage-free refinement, $(\Omega_i^*)_{i\in I}$, of $(\Omega_i)_{i\in I}$, such that $\underline{\boldsymbol{\Omega}} \subset \Omega_i^* \subset \Omega_i$, for each $i \in I$, from observing arbitrage opportunities on portfolios. They may then elect any consistent beliefs, $(\pi_i^*)_{i\in I}$. Throughout, we refer to $(\Omega_i^*, \pi_i^*)_{i\in I}$, or $(\Omega_i^*)_{i\in I}$, as the final information structure. Moreover, $(\Omega_i^*)_{i\in I}$=$(\Omega_i)_{i\in I}$ if and only if $(\Omega_i)_{i\in I}$ is arbitrage-free at the outset. We now study what information markets may reveal.

3 Information markets may reveal

To simplify exposition, anticipation sets, Ω_i (for each $i \in I$), are, at first, finite and we let $S := \cup_{i\in I} \Omega_i$ and $S^* := \cup_{i\in I} \Omega_i^*$. State prices replace mappings in Claim 1-(ii).

We define (for some $J^* \leqslant J$, with a slight abuse in notations) the $S \times J$ and $S^* \times J^*$ matrixes, $V := (V(\omega)) := (v_j(\omega))_{j \in \{1,...,J\}, \omega \in S}$ and $V^* := (V^*(\omega)) := (v_j(\omega))_{j \in \{1,...,J^*\}, \omega \in S^*}$, from the payoff mapping of Section 2, by costlessly eliminating redundant assets, and:

- $Z_\omega := \{\ z \in \mathbb{R}^J :\ V(\omega) \cdot z = 0\ \}$, for each $\omega \in S$, and $Z_\omega^\perp$ its orthogonal;
- $Z_\omega^* := \{\ z \in \mathbb{R}^{J^*} :\ V^*(\omega) \cdot z = 0\ \}$, for each $\omega \in S^*$, and $Z_\omega^{*\perp}$ its orthogonal;
- $Z_i := \cap_{\omega \in \mathbf{\Omega}_i} Z_\omega$, for each $i \in I$, $Z = \sum_{i \in \mathbf{I}} Z_i$, and their orthogonals, $Z_i^\perp$, $Z^\perp = \cap_{i \in \mathbf{I}} Z_i^\perp$;
- $\underline{Z} = \cap_{\omega \in \underline{\mathbf{\Omega}}} Z_\omega$, $\underline{Z}^* = \cap_{\omega \in \underline{\mathbf{\Omega}}} Z_\omega^*$, and their orthogonals, $\underline{Z}^\perp$ and $\underline{Z}^{*\perp}$;
- $Z_i^* := \cap_{\omega \in \mathbf{\Omega}_i^*} Z_\omega^*$, for each $i \in I$, $Z^* = \sum_{i \in \mathbf{I}} Z_i^*$ and their orthogonals, $Z_i^{*\perp}$, $Z^{*\perp}$;
- W_ω, the straight line of $\mathbb{R}^J$, generated by $w_\omega := (v_j(\omega))_{j \in \{1,...,J\}}$, for each $\omega \in S$;
- $\underline{W} := \sum_{\omega \in \underline{\mathbf{\Omega}}} W_\omega$ and their orthogonals, $W_\omega^\perp$ and $\underline{W}^\perp$.
- We similarly define W_ω^*, w_ω^*, for each $\omega \in S^*$, $\underline{W}^*$ and orthogonals.

Claim 3 *The above vector spaces meet the following Assertions:*

(i) $W_\omega = Z_\omega^\perp$, $\forall \omega \in S$, *and* $W_\omega^* = Z_\omega^{*\perp}$, $\forall \omega \in S^*$;

(ii) $\mathbb{R}^J = \sum_{i \in \mathbf{I}} Z_i^\perp$ *and* $\mathbb{R}^{J^*} = \sum_{i \in \mathbf{I}} Z_i^{*\perp}$;

(iii) $Z \subset \underline{Z} = \underline{W}^\perp$ *and* $Z^* \subset \underline{Z}^* = \underline{W}^{*\perp}$;

(iv) $(\Omega_i^*) = (\Omega_i)$ *if and only if the following condition holds:*

(I) $\nexists (z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0$, $V(\omega_i) \cdot z_i \geqslant 0$, $\forall (i, \omega_i) \in I \times \Omega_i$, *with a strict inequality*;

(v) If $\underline{Z} = \{0\}$, *then,* $(\Omega_i^*) = (\Omega_i)$, *i.e.,* (Ω_i) *is non-revealing (or arbitrage-free);*

(vi) If $I \neq I' = \{i \in I : \Omega_i \neq \underline{\mathbf{\Omega}}\}$, $(\Omega_i^*) = (\Omega_i)$ *if and only if the below condition holds:*

(II): $\nexists (i, z) \in I' \times \underline{Z} : V(\omega) \cdot z \geqslant 0$, $\forall \omega \in \Omega_i$, *with at least one strict inequality;*

(vii) Assume that $I \neq I'$, $\underline{Z} \neq \{0\}$ *and, costlessly for some* $J_1 \leqslant J$, *that the first* J_1^{th} *assets yield a Hamel basis of* $\underline{Z}$. *If* $\{v_j(\omega)\}_{j \in \{1,...,J_1\}, \omega \in S \setminus \underline{\mathbf{\Omega}}} \subset \mathbb{R}_+$, *then,* $Z^* = \underline{Z}^* = \{0\}$ *and, moreover,* (Ω_i) *is fully-revealing if* $\{0\} \neq \{v_j(\omega)\}_{j \in \{1,...,J_1\}}$, *for each* $\omega \in S \setminus \underline{\mathbf{\Omega}}$.

Proof (i) Let $\omega \in S$ be given. If $w_\omega = 0$, then $W_\omega = Z_\omega^\perp = \{0\}$. If $w_\omega \neq 0$, the spaces, W_ω and $Z_\omega^\perp$, are 1-dimensional and contain w_ω, i.e., coincide. The rest is alike. $\square$

(ii) The relations $(\sum_{i\in\mathbf{I}} Z_i^\perp)^\perp = \cap_{i\in\mathbf{I}} Z_i = \{0\}$ and $(\sum_{i\in\mathbf{I}} Z_i^{*\perp})^\perp = \{0\}$ hold, from the elimination of redundant assets, hence, $\mathbb{R}^J = \sum_{i\in\mathbf{I}} Z_i^\perp$ and $\mathbb{R}^{J^*} = \sum_{i\in\mathbf{I}} Z_i^{*\perp}$ hold. $\square$

(iii) From the above definitions and Assertion (i), the relations $Z^\perp = (\sum_{i\in\mathbf{I}} Z_i)^\perp = \cap_{i\in I} Z_i^\perp = \cap_{i\in I}(\sum_{\omega\in\mathbf{\Omega}_i} Z_\omega^\perp) = \cap_{i\in I}(\sum_{\omega\in\mathbf{\Omega}_i} W_\omega) \supset \underline{W} = \underline{Z}^\perp$ hold. Assertion (iii) follows. $\square$

(iv) Assertion (iv) states the AFAO characterization of Claim 1, above, in the finite dimensional case, proved directly in Cornet-De Boisdeffre (2002, p. 401). $\square$

(v) Assume that $\underline{Z} = \{0\}$ and, by contraposition, that (Ω_i) fails to be arbitrage-free. From Assertion (iv), there exists $(z_i) \in (\mathbb{R}^J)^I \backslash \{0\}$, such that $\sum_{i=1}^m z_i = 0$ and $V(\omega) \cdot z_i \geqslant 0$ for every pair $(i, \omega) \in I \times \mathbf{\Omega}$. These joint relations imply $V(\omega) \cdot z_i = 0$ for every $(i, \omega) \in I \times \mathbf{\Omega}$, that is, $(z_i) \in \underline{Z}^m \backslash \{0\}$, contradicting the fact that $\underline{Z} = \{0\}$. This contradiction proves that (Ω_i) is arbitrage-free, i.e., from Section 2, $(\Omega_i^*) = (\Omega_i)$. $\square$

(vi) Assume, by contraposition, that $I \neq I'$, $(\Omega_i^*) = (\Omega_i)$ and Condition (II) of Assertion (vi) fails. Then, there exists $(i, z) \in I' \times \underline{Z}$, such that $V(\omega) \cdot z \geqslant 0$ for all $\omega \in \Omega_i$ and $\sum_{\omega\in\Omega_i} V(\omega) \cdot z > 0$. One agent, say $j \in I \backslash I'$ is fully informed. Then, Condition (I) of Assertion (iv) fails with $(z_i, z_j) = (z, -z)$, that is, (Ω_i) fails to be arbitrage-free, which contradicts the above relation, $(\Omega_i^*) = (\Omega_i)$. This contradiction shows the relation $(\Omega_i^*) = (\Omega_i)$ implies Condition (II) to hold. Assume, now, that $(\Omega_i^*) \neq (\Omega_i)$. From Assertion (iv) and the proof of Assertion (v), there exists $(z_i) \in (\underline{Z})^m$, such that $V(\omega_i) \cdot z_i \geqslant 0$ for each $(i, \omega_i) \in I \times \Omega_i$, with one strict inequality, hence, Condition (II) fails. This proves that Condition (II) implies the relation $(\Omega_i^*) = (\Omega_i)$ to hold. $\square$

(vii) Assertion (vii) stems from Assertion (vi) and redundant asset elimination. $\square$

We now extend Claim 3 to the general setting, eased by the fact that all vector spaces, defined below, are finite dimensional, hence, have orthogonal supplements.

We let $S :=\cup_{i\in I}\Omega_i$, $S^*:=\cup_{i\in I}\Omega_i^*$ and derive the mappings $V : \omega \in S \mapsto V(\omega) := (v_j(\omega))_{j\in\{1,...,J\}}$ and V^*: $\omega \in S^* \mapsto V(\omega) := (v_j(\omega))_{j\in\{1,...,J^*\}}$, from the one in Section 2, where we obtain J and $J^* \leqslant J$ by eliminating redundant assets, if any.

For each agent, $i \in I$, and every state, ω, in S or S^*, we define, in the general model, the vector spaces, Z_ω, Z_ω^*, W_ω, W_ω^*, Z_i, Z_i^*, Z, $\underline{Z}$, Z^*, $\underline{Z}^*$ and their orthogonals, in the same way as above (for the finite economy), and, similarly, the vector spaces, $\underline{W} := \{ z \in \mathbb{R}^{J^*} : \exists f \in L_2(\pi)$, *the Riesz space, such that* $z = \int_{\omega\in\mathbf{\Omega}} V(\omega)f(\omega)d\pi(\omega) \}$ and $\underline{W}^*$, for any belief, π, having $\mathbf{\Omega}$ for support, and their orthogonals, $\underline{W}^\perp$ and $\underline{W}^{*\perp}$.

Claim 3 may be restated, as is, in the general model, relative to the latter vector spaces. We let the reader check that all its Assertions hold, mutatis mutandis, and have similar proofs, in the general model, because all above defined vector spaces, or orthogonals, have a finite Hamel basis, in either sets $\{V(\omega)\}_{\omega\in S}$ or $\{V^*(\omega)\}_{\omega\in S^*}$. $\square$

4 Conclusion

Claim 3 and its extension to the general model show that the information markets may reveal depends crucially on the span of assets' payoffs in commonly expected states. Thus, if $\underline{W} = \underline{Z}^\perp = \mathbb{R}^J$, financial markets are non-revealing. In economies where real assets are exchanged and agents have many common forecasts (including price forecasts), markets are, thus, typically non-informative (with $\underline{W} = \mathbb{R}^J$). Contrarily, financial markets insuring primarily idiosyncratic risks (with $\underline{W} \neq \mathbb{R}^J$), would typically be fully revealing, if one agent has full information (along Claim 4-(vii)), or partially revealing otherwise. In particular, in the De Boisdeffre (2016, [4]) model, markets would be non-revealing (with $\underline{W} = \mathbb{R}^J$), because the set of common forecasts, $\underline{\mathbf{\Omega}}$, and the span of payoffs are typically large.

In the latter economy, equilibrium exists - whatever the beliefs, $(\pi_i^*)_{i\in I}$ - if the set $\underline{\mathbf{\Omega}} := \cap_{i\in I}\ \Omega_i^*$ includes a so-called "*minimum incertainty set*", Δ, which features the uncertainty agents could face on future prices, because their forecasts and beliefs are all private. This set, Δ, embeds, among others, all stantard (normalized) perfect foresight equilibrium prices, when they exist. Studying its cardinality, would certainly provide additional insights on the information that markets may convey.

References

[1] Cornet, B., De Boisdeffre, L.: Arbitrage and price revelation with asymmetric information and incomplete markets. Journal of Math. Econ. 38, (2002).

[2] De Boisdeffre, L.: No-arbitrage equilibria with differential information: an existence proof. Economic Theory 31, 255-269 (2007).

[3] De Boisdeffre, L.: Learning from arbitrage. Economic Theory Bulletin 4(1), 111-119 (2016).

[4] De Boisdeffre, L.: Price revelation and existence of financial equilibrium with incomplete markets and private beliefs. Book in Honor of B. Cornet, Springer (2016).

[5] Radner, R.: Existence of equilibrium plans, prices and price expectations in a sequence of markets. Econometrica 40, 289-303 (1972).

[6] Radner, R.: Rational expectations equilibrium: generic existence and the information revealed by prices. Econometrica 47, 655-678 (1979).